

# HARMONIC MORPHISMS AND THE JACOBI OPERATOR

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ABSTRACT. We prove that harmonic morphisms preserve the Jacobi operator along harmonic maps. We apply this result to prove infinitesimal and local rigidity (in the sense of Toth) of harmonic morphisms to a sphere.

## 1. HARMONIC MORPHISMS

*Harmonics maps*  $\phi : (M, g) \rightarrow (N, h)$  between two smooth Riemannian manifolds are critical points of the energy functional  $E(\phi, \Omega) = \frac{1}{2} \int_{\Omega} |d\phi|^2 dv_g$  for any compact domain  $\Omega \subseteq M$  [4], i.e. the first variation of the energy vanishes for any smooth variation of  $\phi$ . The Euler-Lagrange equation for the energy is the vanishing of the tension field  $\tau_{\phi} = \text{trace } \nabla d\phi$ , where  $\nabla$  denotes the connection on  $T^*M \otimes \phi^{-1}TN$  induced from the Levi-Civita connections  $\nabla^M$  on  $M$  and  $\nabla^N$  on  $N$ . If  $\{e_i\}_{i=1}^m$  is a local orthonormal frame on  $M$  we have  $\tau_{\phi} = \sum_{i=1}^m \{ \nabla_{e_i}^{\phi} (d\phi(e_i)) - d\phi(\nabla_{e_i}^M e_i) \}$  where  $\nabla^{\phi}$  denotes the pull-back connection on  $\phi^{-1}TN$ .

Let  $\phi : (M, g) \rightarrow (N, h)$  be a smooth map between two Riemannian manifolds. The tangent space at a point  $x \in M$  can be decomposed as  $T_x M = H_x \oplus V_x$  where  $V_x = \ker(d\phi_x)$  and  $H_x = V_x^{\perp}$ . The spaces  $V_x$  and  $H_x$  are called the *vertical* and *horizontal space* at the point  $x \in M$ , respectively.

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*Date:* October 1999.

*1991 Mathematics Subject Classification.* 58E20.

*Key words and phrases.* Harmonic maps, harmonic morphisms, Jacobi operator, rigidity.

Let  $C_\phi = \{x \in M \mid \text{rank}(d\phi_x) \text{ is not maximal}\}$ ; points of  $C_\phi$  are called *critical points* of  $\phi$ .

**Definition 1.1.** A map  $\phi : (M, g) \rightarrow (N, h)$  is called *horizontally (weakly) conformal* if, for every  $x \in M$ , either  $d\phi_x|_{H_x}$  is conformal and surjective or  $d\phi_x = 0$ .

In particular, either  $x \in M \setminus C_\phi$  and  $\phi$  is submersive at  $x$  or  $x \in C_\phi$  and the differential  $d\phi_x$  has rank 0. If  $\phi : M \rightarrow N$  is horizontally (weakly) conformal, then there exists a function  $\lambda : M \setminus C_\phi \rightarrow \mathbb{R}^+$  such that  $h_{\phi(x)}(d\phi_x(X), d\phi_x(Y)) = \lambda^2 g_x(X, Y)$ , for all  $X, Y \in H_x$ , and  $x \in M \setminus C_\phi$ . The function  $\lambda$  can be extended continuously to the whole of  $M$  by setting  $\lambda|_{C_\phi} = 0$ ; the extended function is called the *dilation function* of  $\phi$ . Note that  $\lambda^2$  is smooth.

*Harmonic morphisms* between Riemannian manifolds are defined to be maps which pull back harmonic functions to harmonic functions; they are characterized as the harmonic maps which are horizontally (weakly) conformal [6, 10]. Note that non-constant harmonic morphisms can only exist if  $\dim M \geq \dim N$ . For a bibliography of papers on harmonic morphisms and an ‘atlas’ of the known examples, see [7]; for the general theory, see [2].

## 2. SECOND VARIATION AND THE JACOBI OPERATOR

Let  $\phi : M \rightarrow N$  be a harmonic map, for simplicity we assume that  $M$  is compact. Then the first variation  $D_V E(\phi) = 0$  for all vector fields along  $\phi$ , where by a *vector field along  $\phi$*  we mean a smooth section  $V \in \Gamma(\phi^{-1}TN)$  of  $\phi^{-1}TN$ . Given a vector field along  $\phi$  we consider a smooth one-parameter variation  $\phi_t$  ( $-\epsilon < t < \epsilon$ ) of  $\phi = \phi_0$  such that  $V = \frac{d}{dt}(\phi_t)|_{t=0}$ . Let  $\nabla^2 = \nabla \circ \nabla^\phi$  so that

$$\nabla_{X,Y}^2 W = \nabla_X^\phi(\nabla_Y^\phi W) - \nabla_{\nabla_X^M Y}^\phi W \quad (X, Y \in TM, W \in \Gamma(\phi^{-1}TN));$$

note that this is tensorial in  $X$  and  $Y$  but not, in general, symmetric. We have the following second variation formula for the energy [8, 12]; we use Milnor's convention for the curvature as in [5].

$$\begin{aligned} H_\phi(V, V) &= \left. \frac{d^2 E(\phi_t)}{dt^2} \right|_{t=0} = \int_M \langle \Delta^\phi V - \text{trace } R^N(d\phi, V)d\phi, V \rangle v_g \\ &= \int_M \langle J^\phi V, V \rangle v_g. \end{aligned}$$

Here  $\Delta^\phi$  is the Laplacian on sections of  $\phi^{-1}TN$  given in a local orthonormal frame  $\{e_i\}$  on  $M$  by

$$\Delta^\phi = -\text{trace } \nabla^2 = -\sum_{i=1}^m \nabla_{e_i, e_i}^2 = -\sum_{i=1}^m \{ \nabla_{e_i}^\phi \nabla_{e_i}^\phi - \nabla_{\nabla_{e_i}^\phi e_i}^\phi \}.$$

The operator  $J^\phi = \Delta^\phi - \text{trace } R^N(d\phi, \cdot)d\phi$  is called the *Jacobi operator* of  $\phi$ ; the elements of its kernel are called *Jacobi fields* and form a subspace  $J(\phi) \subset \Gamma(\phi^{-1}TN)$ .

**Definition 2.1.** The *index* of  $\phi$  is the dimension of the largest subspace of  $\Gamma(\phi^{-1}TN)$  on which  $H_\phi$  is negative definite. The *nullity* of  $\phi$  is the dimension of the kernel of  $J^\phi$ . Alternatively, the index of  $\phi$  is the sum of the multiplicities of the negative eigenvalues  $\lambda$  of  $J^\phi$  and the nullity of  $\phi$  is the multiplicity of the eigenvalue 0 of  $J^\phi$ .

Note that the Jacobi operator is a linear elliptic self-adjoint operator with positive principal part  $\Delta^\phi$ . It follows from standard elliptic theory (cf. [8]) that the index and the nullity are both finite.

Our main result is that harmonic morphisms preserve the Jacobi operator along harmonic maps as follows:

**Theorem 2.2.** *Let  $\phi : M \rightarrow N$  be a harmonic morphism,  $\psi : N \rightarrow P$  a harmonic map and  $V$  a vector field along  $\psi$ . Then the Jacobi operator for the*

vector field  $V \circ \phi$  along  $\psi \circ \phi$  is given by

$$J^{\psi \circ \phi}(V \circ \phi) = \lambda^2 J^\psi(V) \circ \phi,$$

where  $\lambda$  is the dilation function of  $\phi$ . In particular, if  $V$  is a Jacobi field along  $\psi$ , then  $V \circ \phi$  is a Jacobi field along  $\psi \circ \phi$ .

*Proof.* We first remark that the composition  $\psi \circ \phi$  is harmonic. Set  $W = V \circ \phi \in \Gamma(\phi^{-1}\psi^{-1}TP)$ . By definition of pull-back connection, for any  $X \in \Gamma(TM)$ , we have

$$(2.1) \quad \nabla_X^{\psi \circ \phi} W = \left\{ \nabla_{d\phi(X)}^\psi V \right\} \circ \phi.$$

Let  $\gamma$  be a smooth curve in  $M$  tangent to  $X$  and extend  $X$  to a vector field along  $\gamma$ . Then, by (2.1), both sides of the next equation are well-defined and we have

$$\nabla_X^{\psi \circ \phi} (\nabla_X^{\psi \circ \phi} W) = \left\{ \nabla_{d\phi(X)}^\psi (\nabla_{d\phi(X)}^\psi V) \right\} \circ \phi.$$

Again, by (2.1),

$$\nabla_{\nabla_X^M X}^{\psi \circ \phi} W = \left\{ \nabla_{d\phi(\nabla_X^M X)}^\psi V \right\} \circ \phi$$

so that

$$\begin{aligned} \nabla_{X,X}^2 W &= \nabla_X^{\psi \circ \phi} (\nabla_X^{\psi \circ \phi} W) - \nabla_{\nabla_X X}^{\psi \circ \phi} W \\ &= \left\{ \nabla_{d\phi(X)}^\psi (\nabla_{d\phi(X)}^\psi V) - \nabla_{d\phi(\nabla_X^M X)}^\psi V \right\} \circ \phi. \end{aligned}$$

Since  $\phi$  is harmonic, for any orthonormal frame  $\{e_i\}$  on  $M$  we have

$$\sum_{i=1}^m d\phi(\nabla_{e_i}^M e_i) = \sum_{i=1}^m \nabla_{d\phi(e_i)}^N d\phi(e_i)$$

so that

$$\begin{aligned} \text{trace } \nabla^2 W &= \sum_{i=1}^m \{ \nabla_{d\phi(e_i)}^\psi \nabla_{d\phi(e_i)}^\psi V - \nabla_{\nabla_{d\phi(e_i)}^N d\phi(e_i)}^\psi V \} \circ \phi \\ &= \sum_{i=1}^m \{ \nabla_{d\phi(e_i), d\phi(e_i)}^2 V \} \circ \phi. \end{aligned}$$

Now choosing the bases  $\{e_i\}$ ,  $\{f_i\}$  at the points  $x$ ,  $\phi(x)$  such that  $d\phi(e_i) = \lambda f_i$  for  $i = 1, \dots, n$  and  $d\phi(e_i) = 0$  otherwise, we have

$$\begin{aligned} \text{trace } \nabla^2 W &= \sum_{i=1}^n \nabla_{d\phi(e_i), d\phi(e_i)}^2 V \circ \phi \\ &= \lambda^2 \text{trace } \nabla^2 V \circ \phi. \end{aligned}$$

Next, an easy calculation gives

$$\text{trace } R^P(d(\psi \circ \phi), W) d(\psi \circ \phi) = \lambda^2 \text{trace } R^P(d\psi, V) d\psi$$

so that

$$\begin{aligned} J^{\psi \circ \phi}(W) &= J^{\psi \circ \phi}(V \circ \phi) = \lambda^2 \text{trace} \{ -\nabla^2 V - R^P(d\psi, V) d\psi \} \circ \phi \\ &= \lambda^2 J^\psi(V) \circ \phi. \end{aligned}$$

□

**Remark 2.3.** A version of Theorem 2.2 with the assumption that the fibres of  $\phi$  are minimal was proved by the first author in [9].

**Corollary 2.4.** *Let  $\phi : M \rightarrow N$  be a non-constant harmonic morphism between compact manifolds and let  $\psi : N \rightarrow P$  be a harmonic map. Then*

- (a) (i)  $\text{index}(\psi \circ \phi) \geq \text{index}(\psi)$ ;
- (ii)  $\text{index}(\phi) \geq \text{index}(\text{Id}^N)$ , in particular, if  $\text{Id}^N$  is unstable, then so is  $\phi$ ;

(b) (iii)  $\text{nullity}(\psi \circ \phi) \geq \text{nullity}(\psi)$ ;

(iv)  $\text{nullity}(\phi) \geq \text{nullity}(\text{Id}^N)$ .

*Proof.* (a) For part (i), let  $V \in \Gamma(\psi^{-1}TP)$  be an eigenvector with negative eigenvalue, thus

$$(2.2) \quad J^\psi(V) = \alpha V$$

for some constant  $\alpha < 0$ . Set  $W = V \circ \phi \in \Gamma((\psi \circ \phi)^{-1}TP)$ . Then, by Theorem 2.2,

$$\begin{aligned} J^{\psi \circ \phi}(W) &= \lambda^2 J^\psi(V) \circ \phi \\ &= \lambda^2 \alpha (V \circ \phi) = \lambda^2 \alpha W. \end{aligned}$$

Now  $W$  cannot be identically zero; if it were,  $V$  would be zero on  $\phi(M)$ ; this is an open set since any non-constant harmonic morphism is an open mapping [6]. But then, by the unique continuation theorem [1, 3] applied to (2.2),  $V$  would be identically zero on  $N$ . Further, by unique continuation for harmonic morphisms [6],  $\lambda$  cannot be zero on an open set. Hence

$$H_\phi(W, W) = \int_M \lambda^2 \alpha \langle W, W \rangle < 0.$$

Now let  $V_1, \dots, V_s$  be linearly independent eigenvectors with negative eigenvalues. Then, again by unique continuation,  $W_1 = V_1 \circ \phi, \dots, W_s = V_s \circ \phi$  are linearly independent. Further, if  $V_i$  and  $V_j$  correspond to different eigenvalues  $\alpha_i$  and  $\alpha_j$ ,

$$\alpha_i \int_M \lambda^2 \langle W_i, W_j \rangle = \int_M \langle J^{\psi \circ \phi} W_i, W_j \rangle = \int_M \langle W_i, J^{\psi \circ \phi} W_j \rangle = \alpha_j \int_M \lambda^2 \langle W_i, W_j \rangle$$

so that  $H_{\psi \circ \phi}(W_i, W_j) = 0$ . It follows that  $H_\phi$  is negative definite on the span of the  $W_i$  and the estimate on the index follows. Part (ii) follows by putting  $\psi = \text{Id}^N$ .

(b) Let  $V$  be a Jacobi vector field along  $\psi$ . Then, by Theorem 2.2, so is  $W = V \circ \phi$  and we argue as in part (a).  $\square$

### 3. TOTH RIGIDITY OF HARMONIC MORPHISMS

Let  $\phi : M \rightarrow N$  be a harmonic map. A section  $V \in \Gamma(\phi^{-1}TN)$  is said to be a *harmonic variation* of  $\phi$  if  $\phi_t = \exp(tV) : M \rightarrow N$  is harmonic for all  $t \in \mathbb{R}$ . Let  $H(\phi) \subset \Gamma(\phi^{-1}TN)$  denote the set of all harmonic variations of a given map  $\phi : M \rightarrow N$ .

Note that, if  $\phi_t$  is a variation of  $\phi$  through harmonic maps, then  $V = \frac{\partial \phi_t}{\partial t}|_{t=0}$  is a Jacobi field along  $\phi$  [5]. The converse is not always true, in fact there are examples of Jacobi fields along a harmonic map which do not arise as variations through harmonic maps (see, e.g., [11]). Nevertheless we have the following.

**Theorem 3.1** ([13]). *Let  $\phi : (M, g) \rightarrow N = N(c)$  be a harmonic map from a compact Riemannian manifold to a real space form of constant curvature  $c \neq 0$ . Then a section  $V \in \Gamma(\phi^{-1}TN)$  is a harmonic variation if and only if  $V$  is a Jacobi field with  $\|V\|$  constant and  $\text{trace}\langle d\phi, \nabla^\phi V \rangle = 0$ .*

Let  $\phi : M \rightarrow \mathbb{S}^n$  be a harmonic map to the Euclidean sphere of dimension  $n$  ( $n \in \{1, 2, \dots\}$ ) and set

$$K(\phi) = \{V \in J(\phi) : \text{trace}\langle d\phi, \nabla^\phi V \rangle = 0\}.$$

Then  $K(\phi) \subset J(\phi)$  is a linear subspace and, by Theorem 3.1,

$$H(\phi) = \{V \in K(\phi) : \|V\| \text{ is constant} \}.$$

Recall that a variation  $V \in \Gamma(\phi^{-1}TN)$  is called *projectable* if  $\phi(x) = \phi(x')$  implies that  $V(x) = V(x')$ .

**Definition 3.2** ([13]). A harmonic map  $\phi : M \rightarrow \mathbb{S}^n$  is said to be

1. *infinitesimally rigid* if, for every projectable  $V \in K(\phi)$ , there exists  $X \in so(n+1)$  such that the equation  $V = X \circ \phi$  holds;
2. *locally rigid* if, for every projectable harmonic variation  $V$ , there exists a 1-parameter subgroup  $(g_t) \subset O(n+1)$  such that  $\phi_t = \exp(tV) = g_t \circ \phi$  for all  $t \in \mathbb{R}$ .

**Lemma 3.3** ([13]). *Let  $\text{Id}^{\mathbb{S}^n} : \mathbb{S}^n \rightarrow \mathbb{S}^n$  be the identity map. Then*

$$K(\text{Id}^{\mathbb{S}^n}) = so(n+1).$$

We extend a theorem of Toth [13] on rigidity of harmonic Riemannian submersions to submersive harmonic morphisms.

**Theorem 3.4.** *(i) Any surjective and submersive harmonic morphism  $\phi : M \rightarrow \mathbb{S}^n$  is infinitesimally rigid.*

*(ii) Let  $n$  be odd. Then any surjective and submersive harmonic morphism  $\phi : M \rightarrow \mathbb{S}^n$  is locally rigid.*

*Proof.* (i) Let  $V \in K(\phi)$  be projectable. Then since  $\phi$  is surjective and submersive we have  $V = X \circ \phi$  for some  $X \in \Gamma(T\mathbb{S}^n)$  and so, from Theorem 2.2,

$$(3.1) \quad J^\phi(V) = \lambda^2 J^{\mathbb{S}^n}(X) \circ \phi,$$

where  $J^{\mathbb{S}^n}$  denotes the Jacobi operator along  $\text{Id}^{\mathbb{S}^n}$ . Moreover, choosing orthonormal bases  $\{e_i\}$  at a point  $x \in M$  and  $\{f_i\}$  at  $\phi(x) \in N$  such that  $d\phi(e_i) = \lambda f_i \circ \phi$  for  $i = 1, \dots, n$  and  $d\phi(e_i) = 0$  otherwise, we have

$$\begin{aligned}
 (3.2) \quad 0 = \text{trace}\langle d\phi, \nabla^\phi V \rangle &= \sum_{i=1}^m \langle d\phi(e_i), \nabla_{e_i}^\phi (X \circ \phi) \rangle \\
 &= \lambda^2 \sum_{i=1}^n \langle f_i, \nabla_{f_i} X \rangle \circ \phi \\
 &= \lambda^2 \text{trace}\langle d(\text{Id}^{\mathbb{S}^n}), \nabla X \rangle \circ \phi
 \end{aligned}$$



where  $\nabla$  denoted the Levi-Civita connection on  $S^n$ . Equations (3.1) and (3.2) imply that  $X \in K(\text{Id}^{\mathbb{S}^n})$  and by Lemma 3.3,  $X \in so(n+1)$ . This proves that  $\phi$  is infinitesimally rigid.

(ii) Suppose that  $V \in H(\phi)$  so that  $\|V\|$  is constant. As in part (i),  $V = X \circ \phi$  for some  $X \in \Gamma(TS^n)$ . Since  $\phi$  is surjective,  $\|X\|$  is also constant and, since, as in the first part of the proof,  $X$  is Killing, for any vector field  $Y$  on  $\mathbb{S}^n$  we have

$$\langle Y, \nabla_X X \rangle = -\langle X, \nabla_Y X \rangle = -\frac{1}{2}Y\|X\|^2 = 0.$$

It follows that  $\nabla_X X = 0$  on  $\mathbb{S}^n$  which implies that every integral curve  $t \rightarrow g_t(x)$  ( $t \in \mathbb{R}$ ,  $x \in \mathbb{S}^n$ ) of  $X$  is a geodesic. Hence

$$\phi_t(x) = \exp(tV_x) = \exp(tX_{\phi(x)}) = g_t(\phi(x))$$

for all  $x \in M$ , and, since  $X \in so(n+1)$ , we have  $(g_t) \subset O(n+1)$ .

□

We remark that part (ii) is true for  $n$  even but says nothing new as there are no non-trivial projectable harmonic variations  $V \in K(\phi)$  in that case.

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